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Alternate-Wrapped Lindley Distribution

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Abstract

Lindley distribution has been a popular distribution, and many of its generalizations are well established in the literature. The wrapped Lindley distribution has also been a good alternative to the existing circular models. Looking at the utility and applicability of the Lindley distribution, in this paper, we introduce a new circular distribution, namely the alternate-wrapped Lindley (AWL) distribution. This distribution is generated by using the alternate wrapping method (Joshi and Rattihalli (2022)) on the Lindley distribution. The properties of the AWL model are derived, and the maximum likelihood estimator of the model parameter is obtained. To check the performance of the estimator, a simulation study is performed. The proposed model is then compared with other circular models with the help of real data analysis.

Keywords: Akaike Information Criterion (AIC); Alternate Wrapped Exponential (AWE) Distribution; Lindley Distribution; Rose Diagram; Wrapped Lindley (WL) Distribution

MSC Classification: 60E05 , 62E10.

1 Introduction

Circular distributions have been used widely to model angular data arising from many real-life situations like the vanishing angle of birds from the horizon, the orientation of the animal, time to event measured round the clock, and many more (Jammalmadaka and Sengupta (2001)). Various circular models have been developed in the literature to analyze angular data. For instance, Jammalmadaka and Kozubowski (2004) proposed a new family of circular distributions to model skew circular data. (Rao et al. (2007) obtained the wrapped versions of some life-testing models and discussed their properties. Roy and Adnan (2012) derived the wrapped version of weighted exponential

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distribution and discussed its applications. Jacob and Jayakumar (2013) considered wrapped geometric distribution. Joshi and Jose (2018) proposed WL distribution and compared the model with the existing wrapped exponential (WE) model. Yilmaz and Bicer (2018) introduced transmuted wrapped exponential distribution by using the Transmutation Rank-Map method. They showed that the new model is more flexible than the conventional WE model. Christophe et al. (2021) introduced wrapped the modified Lindley distribution and discussed its applications to two real-life data sets.

Recently, Joshi and Rattihalli (2022) introduced a new wrapping technique "alternate wrapping" to generate new circular models and studied the AWE and alternate-wrapped normal (AWN) distributions. In alternate wrapping, the linear density is wrapped around a unit circle first in an anti-clockwise direction. After one wrap, the wrapping direction is changed to clockwise, and this process is continued like this. Joshi and Rattihalli (2022) discussed many interesting properties of alternate wrapping, which motivates to develop more alternate-wrapped circular distributions.

In this paper, we introduce AWL distribution and the model is derived in Section 2. Some of the properties of AWL model are discussed in Section 3. The estimation of the parameter of AWL model is discussed in Section 4 followed by a simulation study in Section 5. Section 6 consists of a real-life data analysis using AWL model and its comparison with the existing circular models. Lastly, Section 7 concludes the findings.

2 AWL Distribution

Let X follow Lindley distribution with probability density function

$$f(x) = \frac{\lambda^2}{1+\lambda} e^{-\lambda x}(1+x), \quad x > 0, \lambda > 0. \tag{1}$$

The alternate wrapped Lindley model is obtained by wrapping (1) in alternate directions (Joshi and Rattihalli (2022)). The first wrap is done in the anti-clockwise direction followed by the next in the clockwise direction. Let θ be the alternate wrapped Lindley variable, then the probability density function of θ is given by

$$g_{aw}(\theta) = f(\theta) + \sum_{m=1}^{\infty} \left(f(4m\pi - \theta) + f(4m\pi + \theta) \right), \quad \theta \in [0, 2\pi).$$

This implies for $\theta \in [0, 2\pi)$,

$$g_{aw}(\theta) = \frac{\lambda^2 e^{-\theta\lambda} \left((-\theta + 4\pi - 1)e^{4\pi\lambda} + (-\theta + 4\pi + 1)e^{2(\theta+2\pi)\lambda} + (\theta - 1)e^{2\theta\lambda} + (\theta + 1)e^{8\pi\lambda} \right)}{\left(e^{4\pi\lambda} - 1\right)^2 (\lambda + 1)}. \tag{2}$$

Remark 1. $\int_0^{2\pi} g_{aw}(\theta) d\theta = 1$.
The linear and circular representation of AWL density is given in Figure 1.

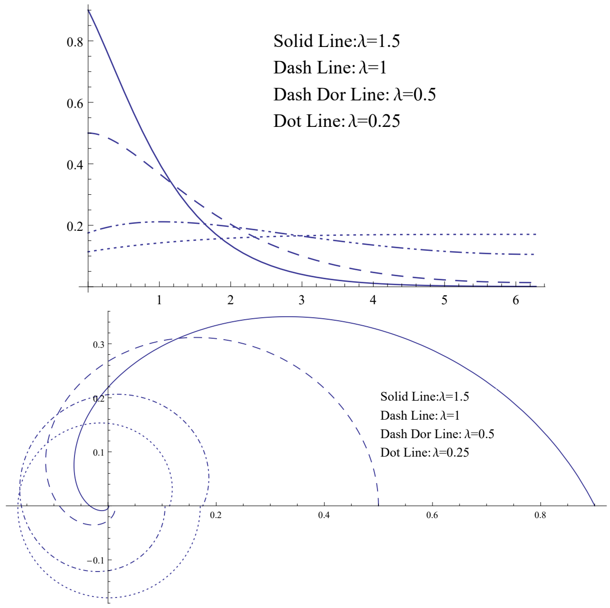


Figure 1: Linear and circular representation of AWL density.

3 Properties of AWL Distribution

In this section, we derive trigonometric moments and other properties of AWL model.

3.1 Trigonometric Moments

In this section we find the trigonometric moments.

$$
\begin{aligned}
\alpha_p &= E(\cos p\theta) \\
&= \frac{\lambda^2\left(\left(e^{4\pi\lambda}-1\right)^2\left(\lambda^2(\lambda+1)+(\lambda-1)p^2\right)+4e^{4\pi\lambda}p\sin(2\pi p)\left(2\pi\cosh(2\pi\lambda)\left(\lambda^2+p^2\right)\right)\right)}{\left(e^{4\pi\lambda}-1\right)^2(\lambda+1)\left(\lambda^2+p^2\right)^2} \\
&+ \frac{\lambda^2\left(\sinh(2\pi\lambda)\left(\lambda(\lambda+2)+p^2\right)\right)}{\left(e^{4\pi\lambda}-1\right)^2(\lambda+1)\left(\lambda^2+p^2\right)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_p = E(\sin p\theta) &= \frac{p\lambda^2\cos(2\pi p)\left(-\lambda^2-2\lambda-p^2+8e^{4\pi\lambda}\pi(p^2+\lambda^2)\right)}{\left(e^{4\pi\lambda}-1\right)^2(\lambda+1)\left(\lambda^2+p^2\right)^2} \\
&+ \frac{p\lambda^2\cos(2\pi p)\left(2e^{2\pi\lambda}\left(2\lambda-(2\pi-1)\left(\lambda^2+p^2\right)-e^{4\pi\lambda}\left(\lambda(2\pi\lambda+\lambda+2)+(1+2\pi)p^2\right)\right)\right)}{\left(e^{4\pi\lambda}-1\right)^2(\lambda+1)\left(\lambda^2+p^2\right)^2}.
\end{aligned}
$$

The p^{th} trigonometric moment ϕ_p, is given by

$$\phi_p = \alpha_p + i\beta_p = \rho_p e^{i\mu_p} = \left(\frac{\lambda^2 \sinh(2\pi\lambda)(i \cosh(2\pi\lambda) - i\cos(2\pi p) + \sin(2\pi p))}{\left(e^{4\pi\lambda} - 1\right)^2 (\lambda+1) \left(\lambda^2 + p^2\right)^2} \right)$$
$$\Bigg(\left(e^{8\pi\lambda} + 1\right) \left(\lambda^2(\lambda+1) + (\lambda-1)p^2\right) + 8\pi e^{4\pi\lambda} p \cosh(2\pi\lambda) \left(\lambda^2 + p^2\right) \sin(2\pi p)$$
$$+ 2e^{4\pi\lambda} \Bigg(- \lambda^2(\lambda+1) + 4\pi i p^3 - 4\pi \imath p \cosh(2\pi\lambda) \left(\lambda^2 + p^2\right) \cos(2\pi p)$$
$$- (\lambda-1)p^2 + 2p \left(\lambda(\lambda+2) + p^2\right) + 4\pi i \lambda^2 p \Bigg)\Bigg).$$

Here, $\rho_p = \sqrt{\alpha_p^2 + \beta_p^2}$ and $\mu_p = \arctan^* \left(\frac{\beta_p}{\alpha_p}\right)$ as defined in Jammalmadaka and Sengupta (2001). The mean direction $\mu_1 = \mu = -\cot^{-1} \left(\frac{2\left(\lambda^3+\lambda^2+\lambda-1\right) \cosh^2(\pi\lambda)}{2\pi(\lambda^2+1)-(\lambda+1)^2 \sinh(2\pi\lambda)} \right)$.
The mean resultant length $\rho_1 = \rho = \sqrt{\frac{\lambda^4\left(\frac{1}{4}((\lambda+1)^2 \sinh(2\pi\lambda) - 2\pi(\lambda^2+1))^2 {}^4(\pi\lambda) + (\lambda^3+\lambda^2+\lambda-1)^2\right)}{(\lambda+1)^2(\lambda^2+1)^4}}$.
The circular variance

$$V_0 = 1 - \rho = 1 - \sqrt{\frac{\lambda^4 \left(\frac{1}{4}\left((\lambda+1)^2 \sinh(2\pi\lambda) - 2\pi\left(\lambda^2+1\right)\right)^2 {}^4(\pi\lambda) + \left(\lambda^3+\lambda^2+\lambda-1\right)^2\right)}{(\lambda+1)^2 \left(\lambda^2+1\right)^4}}.$$

The circular standard deviation σ_0 is given by

$$\sigma_0 = \sqrt{-2\log(1 - V_0)}$$
$$= \sqrt{-\log \left(\frac{\lambda^4 \left(\frac{1}{4}\left((\lambda+1)^2 \sinh(2\pi\lambda) - 2\pi\left(\lambda^2+1\right)\right)^2 \operatorname{sech}^4(\pi\lambda) + \left(\lambda^3+\lambda^2+\lambda-1\right)^2\right)}{(\lambda+1)^2 \left(\lambda^2+1\right)^4} \right)}.$$

4 Estimation of Parameters

Let $\theta_1, \ldots, \theta_n$ be a random sample from (2). Then, the likelihood function is given by

$$L(\lambda|(\underline{\theta}, \underline{\delta})) = \prod_{i=1}^{n} \left(\frac{\lambda^2 e^{-\theta_i\lambda}(-\theta_i + 4\pi - 1)e^{4\pi\lambda}}{\left(e^{4\pi\lambda} - 1\right)^2 (\lambda+1)} \right)$$
$$\prod_{i=1}^{n} \left(\frac{\lambda^2 e^{-\theta_i\lambda} \left((-\theta_i + 4\pi + 1)e^{2(\theta_i+2\pi)\lambda} + (\theta_i - 1)e^{2\theta\lambda} + (\theta+1)e^{8\pi\lambda}\right)}{\left(e^{4\pi\lambda} - 1\right)^2 (\lambda+1)} \right). \tag{3}$$

Therefore, the log-likelihood function is given by

$$\log L(\lambda|\underline{\theta}) = 2n \log \lambda - 2n \log(e^{4\pi\lambda} - 1) - n \log(\lambda + 1) + \sum_{i=1}^{n} \log \Big((-\theta + 4\pi - 1)e^{4\pi\lambda} + (-\theta + 4\pi + 1)e^{2(\theta+2\pi)\lambda} + (\theta - 1)e^{2\theta\lambda} + (\theta + 1)e^{8\pi\lambda}\Big). \quad (4)$$

The likelihood equation for λ can not be solved analytically, and hence we estimate the parameter λ by optimizing (4) numerically by using optimize() function, used for one-dimensional optimization, in R. A simulation study is carried out in Section 5 to study the properties of the maximum likelihood estimator (MLE), $\hat{\lambda}$ of λ.

5 Simulation Study

In this section, we generate samples of sizes 20, 50, 100, and 200 from (2). We obtain the MLE $\hat{\lambda}$ of λ by maximizing the likelihood function (4). Since (4) does not have a closed-form solution, we use the maxLik package in R to obtain the MLE, $\hat{\lambda}$. The simulation results for the bias, mean square error (MSE), and variance (var) of $\hat{\lambda}$ are reported in Table 1. The R code for simulation of data from AWL is given in Appendix. The following results are deduced from Table 1.

Table 1: Average values of bias, mean square error (MSE) and variance (var) for $\hat{\lambda}$ based on 10000 samples

	λ=0.5			λ=1.0		
n	bias($\hat{\lambda}$)	MSE($\hat{\lambda}$)	var($\hat{\lambda}$)	bias($\hat{\lambda}$)	MSE($\hat{\lambda}$)	var($\hat{\lambda}$)
20	-0.011185	0.023207	0.023082	0.026612	0.032284	0.031575
50	-0.005852	0.008674	0.008640	0.011546	0.012773	0.012640
100	-0.001778	0.003620	0.003617	0.005481	0.006234	0.006204
200	-0.000535	0.001638	0.001637	0.002938	0.003016	0.003000
	λ=1.5			λ=2		
20	0.036229	0.064240	0.062927	0.041354	0.095670	0.093959
50	0.020306	0.029196	0.028784	0.026517	0.049984	0.049281
100	0.010423	0.014459	0.014351	0.014618	0.026681	0.026467
200	0.005918	0.007145	0.007110	0.005947	0.012950	0.012915

1. As the sample size increases, the values of bias, MSE, and variance of $\hat{\lambda}$ decrease, showing that the estimator is consistent and asymptotically unbiased.
2. The values of bias, MSE, and variance of $\hat{\lambda}$ increase with the increase in λ.

Additionally, the boxplots of the estimator (Table 4) and the kernel density estimator of the density (Table 5) given in the Appendix, validate that the estimator $\hat{\lambda}$ is asymptotically normal.

6 Real Life Application

In this section, we apply the AWL to a real-life data set. The data set consists of the orientations of 76 turtles after laying eggs (Jammalmadaka and Sengupta (2001)). The data set is given in Table 4. We fit the AWL model on the turtle data and obtain the

Table 2: Direction (in degrees) clockwise from north

8	9	13	13	14	18	22	27	30	34	38	38
40	44	45	47	48	48	48	48	50	53	56	57
58	58	61	63	64	64	64	65	65	68	70	73
78	78	78	83	83	88	88	88	90	92	92	93
95	96	98	100	103	106	113	118	138	153	153	155
204	215	223	226	237	238	243	244	250	251	257	268
285	319	343	350								

estimator. We also fit the AWE distribution, WL distribution, and WE distribution to this data set and compare their performance with that of the AWL model. The estimator along with the standard errors for all the models are given in Table 5. We obtain the AIC and BIC values for all the models for comparison. We also draw the rose diagram (Figure 2) for the given data and plot the densities of all these models on the same graph. Based on the results obtained from AIC, BIC, and rose diagram plot, we conclude that the AWL model performs slightly better than the WE and AWE models and performs as good as the WL model.

Table 3: Comparison of models for turtle data

Model	MLE ($\hat{\lambda}$)	S. E.	log-likelihood	AIC	BIC
AWL	0.7919	0.0726	-119.2425	240.4850	242.8157
WL	0.7482	0.0825	-119.7089	241.4178	243.7485
AWE	0.4800	0.0659	-120.6871	243.3742	245.7049
WE	0.4228	0.0742	-120.6474	243.2948	245.6255

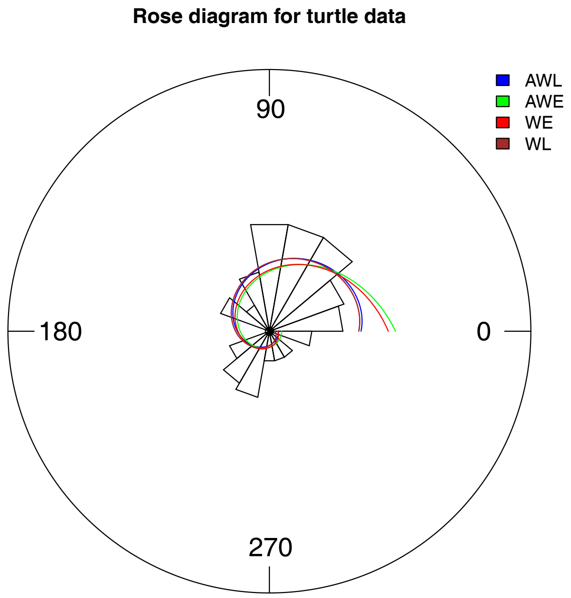


Figure 2: Rose diagram for turtle data

7 Conclusion

This paper introduces a new circular distribution AWL and its properties have been derived. The MLE of the parameter of the proposed distribution is obtained and its properties have been studied through an extensive simulation study. The simulation study along with the box plots and kernel density estimator curves show that the MLE of λ is unbiased, consistent and asymptotically normal. The AWL model is fitted to a real-life data asset and has been compared with AWE, WE, and WL models. Based on the model selection criteria such as AIC, BIC and log-likelihood, it is observed that the AWL model performs slightly better than the AWE and WE models and it performs as well as WL model. The proposed AWL model can be used as a good alternative to the existing circular models in future.

8 Conflict of Interest

The author declares that there is no conflict of interest.

References

Christophe, C., Tomy, L. and Jose, M. (2021). Wrapped modified Lindley distribution. Journal of Statistics and Management Systems, 24(5), 1025-1040.

Jacob, S. and Jayakumar, K. (2013). Wrapped geometric distribution: A new probability model for circular data. Journal of Statistical Theory and Applications, 12(4), 348–355.

Jammalamadaka, S. R. and SenGupta, A. (2001). Topics in Circular Statistics. World Scientific, New York.

Jammalamadaka, S. R. and Kozubowski, T. J. (2004). New families of wrapped distributions for modeling skew circular data. Communications in Statistics-Theory and Methods, 33(9), 2059-2074.

Joshi, S. and Rattihali, R. N. (2022). Alternate-wrapped circular distributions. SORT-Statistics and Operations Research Transactions, 46(2), 245-262.

Joshi, S. and Jose, K. K. (2018). Wrapped Lindley Distribution. Communication in Statistics: Theory and Methods, 47(5), 1013-1021.

Rao, A. V. D., Sarma, I. R. and Girija, S. V. S. (2007). On wrapped version of some life testing models. Communications in Statistics-Theory and Methods, 36, 2027–35.

Roy, S. and Adnan, M. A. S. (2012). Wrapped weighted exponential distributions. Statistics and Probability Letters, 82(1), 77-83.

Yilmaz, A. and Bicer, C. (2018). A new wrapped exponential distribution. Mathematical Sciences, 12, 285-293.

Appendix

Table 4: Box Plots of the 10000 simulated values of MLE $\hat{\lambda}$ for AWL distribution

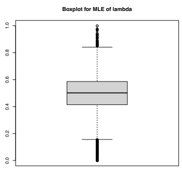
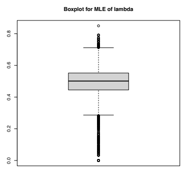
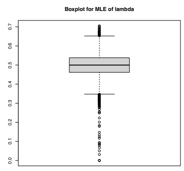
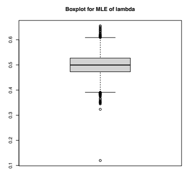
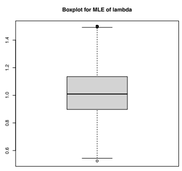
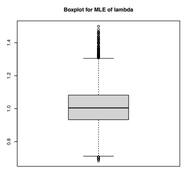
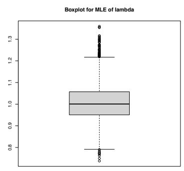
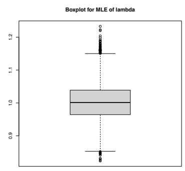
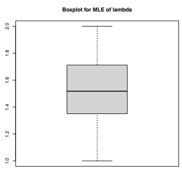
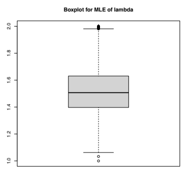
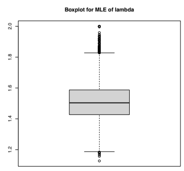
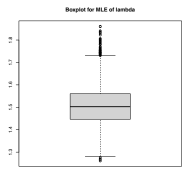
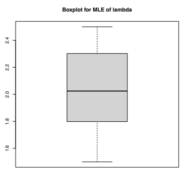
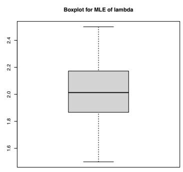
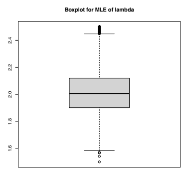
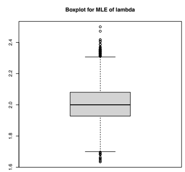
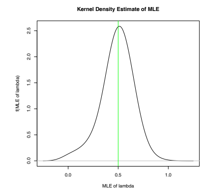
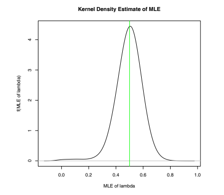
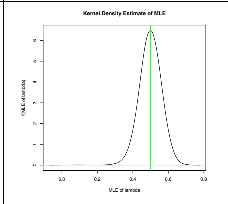
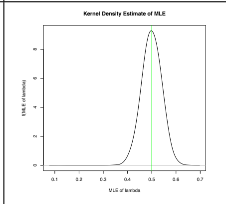
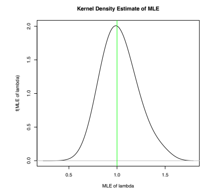
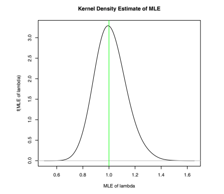
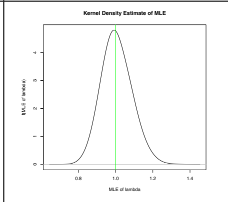
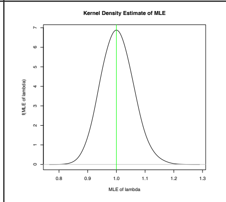
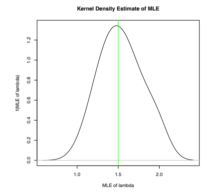
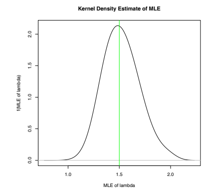
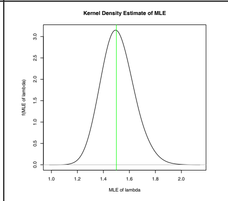
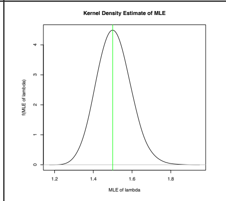
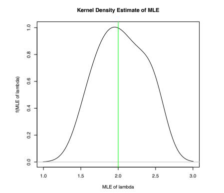
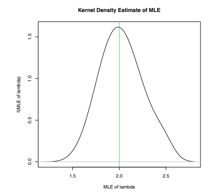
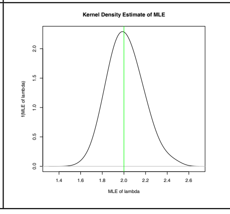
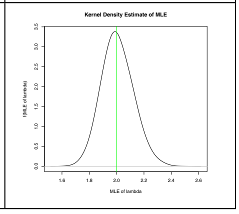
	Box Plots for MLE ($\hat{\lambda}$)			
λ	$n = 20$	$n = 50$	$n = 100$	$n = 200$
0.5				
1				
1.5				
2				

Table 5: Kernel Density Estimator (KDE) of the density of $\hat{\lambda}$ (MLE) based on 10000 simulated values of the estimators of λ for AWL distribution

	KDE for the density of $\hat{\lambda}$			
λ	$n = 20$	$n = 50$	$n = 100$	$n = 200$
0.5				
1				
1.5				
2				

R Code:

```
rm(list=ls(all=TRUE))
library(gsl);library(maxLik)
n1=10000
mledummy=numeric(n1)
for(k in 1:n1)
{
th1=2;m=200;
t=numeric(m)
wl=numeric(m)
u=runif(m,0,1)
for(i in 1:m){
t[i]=(-th1-lambert_Wm1(exp(-th1-1)*(th1+1)*(u[i]-1))-1)/(th1)
}
t
y=t/(4*pi);y
z=y-floor(y);z
wl=c()
for(j   in   1:m){
if(z[j]>0&&z[j]<0.5)
{
wl[j]=z[j]*4*pi
}
if(z[j]>0.5&&z[j]<1)
{
wl[j]=(1-z[j])*4*pi
}
}
wl
n=length(t)
LL=function(lmd){-((2*n*log(lmd))-(n*log(1+lmd))-(lmd*sum(wl))-
(2*n*log(exp(4*pi*lmd)-1))++(sum(log(exp(4*pi*lmd)*(4*pi-1-wl)+
exp(2*(2*pi+wl)*lmd) *(1+4*pi-wl)
+(wl-1)*(exp(2*wl*lmd))+(1+wl)*(exp(8*pi*lmd)))))) }
out = optimize(LL, c(th1-0.5,th1+0.5))
out
mledummy[k]=out$minimum
}
mledummy
mean(mledummy)
bias.lmd=mean(mledummy-th1)
mse.lmd =(1/n1)*sum((mledummy-th1)^2)
var.lmd=(1/n1)*sum((mledummy-mean(mledummy))^2)
bias.lmd;mse.lmd;var.lmd
```